

VARIATION OF GEODESIC LENGTH FUNCTIONS OVER TEICHMÜLLER SPACE

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ABSTRACT. In a family of compact, canonically polarized, complex manifolds equipped with Kähler-Einstein metrics the first variation of the lengths of closed geodesics was previously shown in [A-S] to be the geodesic integral of the harmonic Kodaira-Spencer form. We compute the second variation. For one dimensional fibers we arrive at a formula that only depends upon the harmonic Beltrami differentials. As an application a new proof for the plurisubharmonicity of the geodesic length function and its logarithm (with estimate) follows, which also applies to the previously not known cases of Teichmüller spaces of weighted punctured Riemann surfaces, where the methods of Kleinian groups are not available.

1. INTRODUCTION

In the study of Teichmüller spaces geodesic length functions play an important role, in particular under the aspect of the theory of several complex variables.

In [K] Kerckhoff showed that for a finite number of closed geodesics, which fill up a Riemann surface, the sum of the geodesic length functions provides a proper exhaustion of the corresponding Teichmüller space. In [WO2] Wolpert proved that this function is actually convex along Weil-Petersson geodesics and plurisubharmonic. Later it turned out that the logarithm of a sum of geodesic length functions is plurisubharmonic as well [WO3, WO4]. In [Ye] Yeung constructed a bounded plurisubharmonic exhaustion function together with estimates. The Levi form of the geodesic length functions also played an important role in McMullen's proof of the Kähler hyperbolicity of the moduli space [M].

We want to base our study of geodesic length functions solely upon the hyperbolic geometry of Riemann surfaces and use the methods of Kähler geometry. From this point of view it is desirable to express results in terms of harmonic Beltrami differentials, which are to be considered as harmonic Kodaira-Spencer forms.

This approach avoids entirely methods involving Fuchsian groups. In particular our results extend to cases where uniformization theory is not available, such as Teichmüller and moduli spaces of weighted punctured Riemann surfaces, equipped with conical hyperbolic metrics.

Our methods originate from the study of Kähler-Einstein manifolds of negative curvature, and the computations are done in this framework. Our result for the

first variation of the geodesic length function is surprisingly simple. In dimension one it reads:

Theorem ([A-S, Theorem 3.2]). *Given a holomorphic family of hyperbolic Riemann surfaces $\mathcal{X} \rightarrow S$, the first variation of the length in a family of closed geodesics γ_s is a geodesic integral of the harmonic Beltrami differential*

$$A_i = A_{i\bar{z}} \frac{\partial}{\partial z} \bar{dz}$$

associated to a complex tangent vector $\partial/\partial s^i$, namely

$$\frac{\partial \ell(\gamma_s)}{\partial s^i} = \frac{1}{2} \int_{\gamma_s} A_i.$$

When dealing with tensors of higher order like curvature, which involve second order derivatives of metric tensors, certain integral operators arise in a natural way. In the context of automorphic forms the operator

$$(\square + 1)^{-1},$$

where \square denotes the (complex) Laplacian, was extensively studied (cf. [E]), and Wolpert used it in [WO2]. Later it played a major role in Siu's study of Kähler-Einstein manifolds [SIU] and also in [SCH1, SCH4].

Its counterpart for geodesic integration rather than integration over the whole Riemann surface is the operator

$$\left(-\frac{D^2}{dt^2} + c\right)^{-1}, \quad c = 1, 2,$$

where D/dt denotes covariant differentiation along a geodesic. Our main theorem is the following:

Theorem. *Let $f : \mathcal{X} \rightarrow S$ be a holomorphic family of hyperbolic Riemann surfaces together with a differentiable family of closed geodesics γ_s . Then*

$$\begin{aligned} \frac{\partial^2 \ell(\gamma_s)}{\partial s^i \partial s^{\bar{j}}} &= \frac{1}{2} \left(\int_{\gamma_s} \left((\square + 1)^{-1} (A_i A_{\bar{j}}) + \left(-\frac{D^2}{dt^2} + 2 \right)^{-1} (A_i) \cdot A_{\bar{j}} \right) \right. \\ &\quad \left. + \frac{1}{2\ell(\gamma_s)} \int_{\gamma_s} A_i \cdot \int_{\gamma_s} A_{\bar{j}} \right). \end{aligned}$$

Let $G_{i\bar{j}}^{WP} = \langle A_i, A_{\bar{j}} \rangle_{WP}$ denote the coefficient of the Weil-Petersson metric, and denote by P_1 a certain positive function depending on the diameter (precisely, a lower bound for the resolvent kernel).

Corollary. *The following estimate holds for the second variation:*

$$\begin{aligned} \frac{\partial^2 \ell(\gamma_s)}{\partial s^i \partial s^{\bar{j}}} &\geq \frac{\ell(\gamma_s)}{2} P_1(d(\mathcal{X}_s)) \cdot \langle A_i, A_{\bar{j}} \rangle_{WP} + \frac{1}{2\ell(\gamma_s)} \int_{\gamma_s} A_i \int_{\gamma_s} A_{\bar{j}}, \\ \frac{\partial^2 \log \ell(\gamma_s)}{\partial s^i \partial s^{\bar{j}}} &\geq \frac{1}{2} P_1(d(\mathcal{X}_s)) \cdot \langle A_i, A_{\bar{j}} \rangle_{WP} + \frac{1}{4\ell(\gamma_s)^2} \int_{\gamma_s} A_i \int_{\gamma_s} A_{\bar{j}}. \end{aligned}$$

In particular, $\log \ell(\gamma_s)$ is strictly plurisubharmonic.

(Here " \geq " is used in the sense of Definition 6.3.) Further applications are given in Section 6. We have upper estimates.

Corollary. *Let $\dim S = 1$. Denote by $\|A_s\|_0$ the maximum of the pointwise norm of the harmonic Beltrami differential taken over the fiber \mathcal{X}_s . Then*

$$\begin{aligned} \frac{\partial^2 \ell(\gamma_s)}{\partial s \partial \bar{s}} &\leq \ell(\gamma_s) \|A_s\|_0^2, \\ \frac{\partial^2 \log \ell(\gamma_s)}{\partial s \partial \bar{s}} &\leq \frac{3}{4} \|A_s\|_0^2. \end{aligned}$$

2. FAMILIES OF KÄHLER-EINSTEIN MANIFOLDS

We compute the second variations of the geodesic length function in the general setting of Kähler-Einstein manifolds of negative Ricci curvature.

A Kähler form on a complex manifold X of dimension n will be denoted by

$$\omega_X = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

We use the summation convention together with the ∇ -notation for covariant derivatives. A $|$ -symbol will denote an ordinary derivative. Also, ∂_α and $\partial_{\bar{\beta}}$ will stand for $\partial/\partial z^\alpha$ and $\partial/\partial \bar{z}^\beta$ respectively. The raising and lowering of indices is defined as usual. We also use the semi-colon notation for covariant derivatives. For the Ricci tensor $R_{\alpha\bar{\beta}}$ on X we use the sign convention

$$(1) \quad R_{\alpha\bar{\beta}} = -\log(g(z))_{|\alpha\bar{\beta}},$$

where $g(z) = \det(g_{\alpha\bar{\beta}}(z))$.

Let $\{\mathcal{X}_s\}_{s \in S}$ be a holomorphic family of canonically polarized compact complex manifolds parameterized by a (connected) complex space S . It is given by a proper, smooth, holomorphic mapping $f : \mathcal{X} \rightarrow S$ such that $\mathcal{X}_s = f^{-1}(s)$ for all $s \in S$. For simplicity we will assume that the base S is smooth, although our results can also be given a meaning for possibly non-reduced singular base spaces.

Local coordinates on S will be denoted by s^i , $i = 1, \dots, N$. We use these as local coordinates on the total space \mathcal{X} together with further local coordinates z^α , $\alpha = 1, \dots, n$, where n is the fiber dimension, satisfying $f(z, s) = s$.

The fibers \mathcal{X}_s are equipped with Kähler-Einstein forms

$$\omega_{\mathcal{X}_s} = \sqrt{-1} g_{\alpha\bar{\beta}}(z, s) dz^\alpha \wedge d\bar{z}^\beta$$

of constant negative curvature -1 depending smoothly upon the parameter s . We write $g(z, s) = \det(g_{\alpha\bar{\beta}}(z, s))$ for the family $f : \mathcal{X} \rightarrow S$, where $R_{\alpha\bar{\beta}}(z, s) = -g_{\alpha\bar{\beta}}(z, s)$.

We consider the real $(1, 1)$ -form

$$(2) \quad \omega_{\mathcal{X}} = \sqrt{-1} \partial \bar{\partial} \log g(z, s)$$

on the total space \mathcal{X} . The fiberwise Kähler-Einstein equation (1) implies that

$$\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$$

for all $s \in S$. In particular $\omega_{\mathcal{X}}$, restricted to any fiber, is positive definite. The following fact is known:

Theorem ([SCH4]). *Let $f : \mathcal{X} \rightarrow S$ be nowhere infinitesimally trivial. Then $\omega_{\mathcal{X}}$ is a Kähler form on the total space.*

Let

$$\rho : T_s S \rightarrow H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$$

be the Kodaira-Spencer map for the deformation $f : \mathcal{X} \rightarrow S$ at a point $s \in S$.

The Kähler-Einstein metric $\omega_{\mathcal{X}_s}$ on \mathcal{X}_s induces a natural inner product on the space $H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$ of infinitesimal deformations of \mathcal{X}_s and thus on $T_s S$ via the map ρ ; this is the *Weil-Petersson* Hermitian inner product on $T_s S$. Namely, given tangent vectors $u, v \in T_s S$, we denote by $A_u = A_{u\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$ and A_v the harmonic representatives of $\rho(u)$ and $\rho(v)$ respectively. Then the inner product of u and v equals

$$\langle u, v \rangle_{WP} = \int_{\mathcal{X}_s} A_{u\bar{\beta}}^\alpha \bar{A}_{v\gamma}^{\bar{\delta}} g_{\alpha\bar{\delta}} g^{\bar{\beta}\gamma} g dV,$$

where $A_{\bar{v}}$ denotes the adjoint (conjugate) tensor of A_v , and $g dV$ the volume element.

We note that the Weil-Petersson inner product is positive definite at a given point of the base, if the induced deformation is effective.

We set $A_j = A_{\partial/\partial s^j}$. Then the Weil-Petersson form on S equals

$$\omega^{WP} = \sqrt{-1} G_{i\bar{j}}^{WP}(s) ds^i \wedge d\bar{s}^{\bar{j}},$$

where we use the notation

$$G_{i\bar{j}}^{WP}(s) = \langle \partial/\partial s^i, \partial/\partial s^j \rangle_{WP} = \int_{\mathcal{X}_s} A_{i\bar{\beta}}^\alpha \bar{A}_{j\gamma}^{\bar{\delta}} g_{\alpha\bar{\delta}} g^{\bar{\beta}\gamma} g dV.$$

The short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{X}/S} \rightarrow \mathcal{T}_{\mathcal{X}} \rightarrow f^* \mathcal{T}_S \rightarrow 0$$

induces the Kodaira-Spencer map via the edge homomorphism for direct images. A lift of a tangent vector $\partial/\partial s^i$ at a point s of S is a differentiable vector field on \mathcal{X}_s with values in $\mathcal{T}_{\mathcal{X}}$. It has the form

$$\partial/\partial s^i + b_i^\alpha \partial_\alpha.$$

Its exterior $\bar{\partial}$ -derivative $B_{i\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$, where $B_{i\bar{\beta}}^\alpha = \nabla_{\bar{\beta}} b_i^\alpha$, is interpreted as a $\bar{\partial}$ -closed $(0, 1)$ -form on \mathcal{X}_s with values in the tangent bundle of \mathcal{X}_s . Its cohomology class

$$(3) \quad \rho(\partial/\partial s^i) = [B_{i\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}] \in H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s}).$$

equals the obstruction against the existence of a holomorphic lift of the given tangent vector, i.e. the infinitesimal triviality of the deformation in the direction of the tangent vector.

We now introduce notations that will be used in the rest of the paper.

The *horizontal lift* of $\partial/\partial s^i$, i.e. the lift that is perpendicular to the fibers, will be denoted by

$$(4) \quad v_i = \partial/\partial s^i + a_i^\alpha \partial_\alpha.$$

Note that the quantities a_i^α are in general not tensors. It follows from the definition that

$$(5) \quad a_i^\alpha = -g^{\bar{\beta}\alpha} g_{i\bar{\beta}}.$$

We set

$$(6) \quad A_{i\bar{\beta}}^\alpha = \nabla_{\bar{\beta}} a_i^\alpha.$$

The following properties of the tensors $A_{i\bar{\beta}}^\alpha$ are known (cf. [SCH1, SCH3]) and will be used in the sequel:

Proposition 2.1. *The horizontal lifts of tangent vectors with respect to ω_χ induce the harmonic representatives of Kodaira-Spencer classes in the sense that $A_{i\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$ is the harmonic representative of $\rho(\partial/\partial s^i)$. The coefficients satisfy the following properties*

$$(7) \quad \nabla_{\bar{\delta}} A_{i\bar{\beta}}^\alpha = \nabla_{\bar{\beta}} A_{i\bar{\delta}}^\alpha,$$

$$(8) \quad \nabla_\gamma A_{i\bar{\beta}}^\alpha g^{\bar{\beta}\gamma} = 0,$$

$$(9) \quad A_{i\bar{\beta}\bar{\delta}} = A_{i\bar{\delta}\bar{\beta}}.$$

The conditions (7) and (8) above correspond to harmonicity, whereas condition (9) reflects the relationship with the metric tensor.

We use the notation $\bar{c}^{\bar{\beta}} = \overline{c^\beta}$ for (locally defined) tensors.

Later we will need the following fact:

Lemma 2.2. *The partial derivatives of the Christoffel symbols with respect to the base parameter satisfy the identities*

$$(10) \quad \Gamma_{\gamma\sigma|s^i}^\alpha = -a_{i;\gamma\sigma}^\alpha,$$

$$(11) \quad \Gamma_{\gamma\sigma|\bar{s}^{\bar{j}}}^\alpha = -g^{\bar{\beta}\alpha} a_{\bar{j}\gamma;\bar{\beta}\sigma}.$$

3. FAMILIES OF CLOSED GEODESICS

Let $(f : \mathcal{X} \rightarrow S, \omega_\chi)$ be a family of Kähler-Einstein manifolds with constant negative Ricci curvature -1 , where ω_χ is given by (2).

We denote by γ_s a differentiable family of closed geodesics in the fibers \mathcal{X}_s , and by $\ell(s)$ the length of γ_s . In order to compute first and second variations, it is sufficient to assume that S is a disk in the complex plane centered at 0 with coordinate s (it is even sufficient to assume that the embedding dimension equals one). The general formulas follow from this case by polarization.

In local coordinates (z, s) the closed geodesic curves γ_s are solutions of the differential equation

$$(12) \quad \ddot{u}^\alpha(t, s) + \Gamma_{\gamma\sigma}^\alpha(u(t, s)) \dot{u}^\gamma(t, s) \dot{u}^\sigma(t, s) = 0.$$

The solution is unique up to an affine change of the parameter. In particular we may prescribe any positive constant value of its speed

$$\|\dot{u}(t, s)\|^2 = g_{\alpha\bar{\beta}}(u(t, s), s) \dot{u}^\alpha(t, s) \dot{u}^{\bar{\beta}}(t, s).$$

For $s = 0$ we choose $\|\dot{u}\| = 1$, for the remaining values of s the value of $\|\dot{u}\|$ will be determined by the fact that the parameter t assumes values in the interval $[0, \ell_0]$, where ℓ_0 is the length of γ_0 . Hence the family of geodesics is given by a map

$$(13) \quad u : S \times [0, \ell_0] \rightarrow \mathcal{X}$$

such that $u \circ f$ is the projection onto the first factor. Now

$$(14) \quad u_*(\partial_s) = \partial_s + u_s^\alpha \partial_\alpha + u_s^{\bar{\beta}} \partial_{\bar{\beta}}$$

with partial derivatives

$$u_s^\alpha := u|_s^\alpha \text{ and } u_s^{\bar{\beta}} := (\overline{u^{\bar{\beta}}})|_s.$$

Note that the $(1, 0)$ - and $(0, 1)$ -components $\partial_s + u_s^\alpha \partial_\alpha$ and $u_s^{\bar{\beta}} \partial_{\bar{\beta}}$ of $u_* \partial_s$ are tensors along the geodesics with values in $T\mathcal{X}$ and $T_{\mathcal{X}/S} \subset T\mathcal{X}$ respectively. In a similar way the tensor

$$(15) \quad \dot{u} = u_*(d/dt) = \dot{u}^\alpha \partial_\alpha + \dot{u}^{\bar{\beta}} \partial_{\bar{\beta}}$$

along the family of geodesics has a type decomposition. The difference of two lifts of tangent vectors from the base is a tangent vector along the geodesics (with values in the relative tangent bundle). For $\dim S = 1$ we have the horizontal lift

$$v_s = \partial_s + a^\alpha \partial_\alpha.$$

The difference of $u_*(\partial_s)$ and the horizontal lift has the components

$$(16) \quad u_s^\alpha - a_s^\alpha = u_s^\alpha(s, t) - a_s^\alpha(u(s, t), s),$$

$$(17) \quad u_s^{\bar{\beta}} = u_s^{\bar{\beta}}(s, t).$$

For any tensor along the geodesic γ_s on a fiber \mathcal{X}_s we denote by D/dt the *covariant derivative along γ_s* . In particular

$$(18) \quad \frac{D}{dt} \dot{u} = 0.$$

Let $w^\alpha(t) \partial_\alpha$ be any vector field along γ_s . Then

$$(19) \quad \frac{D}{dt} w^\alpha(t) = \dot{w}^\alpha(t) + \Gamma_{\gamma\sigma}^\alpha(u(t)) w^\gamma(t) \dot{u}^\sigma(t).$$

If $w^\alpha(t)$ is of the form $\tilde{w}^\alpha(u(t))$, then (19) implies

$$(20) \quad \frac{D}{dt} w^\alpha(t) = \tilde{w}^\alpha(u(t))_{;\kappa} \dot{u}^\kappa(t) + \tilde{w}^\alpha(u(t))_{;\bar{\lambda}} \dot{u}^{\bar{\lambda}}(t).$$

Corresponding equations hold for tensors of type $(0, 1)$ and contravariant tensors.

Lemma 3.1. *We have*

$$(21) \quad \frac{D}{dt}(u_s^\alpha - a_s^\alpha) = \dot{u}_s^\alpha + \Gamma_{\gamma\sigma}^\alpha u_s^\gamma \dot{u}_s^\sigma - a_{s;\gamma}^\alpha \dot{u}_s^\gamma - a_{s;\bar{\beta}}^\alpha \dot{u}_s^{\bar{\beta}},$$

$$(22) \quad \begin{aligned} \frac{D^2}{dt^2}(u_s^\alpha - a_s^\alpha) &= \Gamma_{\sigma\mu}^\alpha \Gamma_{\kappa\gamma}^\mu \dot{u}_s^\kappa (\dot{u}_s^\sigma u_s^\gamma - \dot{u}_s^\gamma u_s^\sigma) + \\ &\quad R_{\sigma\gamma\bar{\lambda}}^\alpha \dot{u}_s^\sigma (\dot{u}_s^{\bar{\lambda}}(u_s^\gamma - a_s^\gamma) - u_s^{\bar{\lambda}} \dot{u}_s^\gamma) - \\ &\quad 2A_{s\bar{\delta};\gamma}^\alpha \dot{u}_s^\gamma \dot{u}_s^{\bar{\delta}} - A_{s\bar{\delta};\bar{\tau}}^\alpha \dot{u}_s^{\bar{\delta}} \dot{u}_s^{\bar{\tau}}, \end{aligned}$$

$$(23) \quad \frac{D}{dt}(u_s^{\bar{\beta}}) = \dot{u}_s^{\bar{\beta}} + \Gamma_{\bar{\delta}\bar{\tau}}^{\bar{\beta}} u_s^{\bar{\delta}} \dot{u}_s^{\bar{\tau}},$$

$$(24) \quad \begin{aligned} \frac{D^2}{dt^2}(u_s^{\bar{\beta}}) &= g^{\bar{\beta}\alpha} A_{s\bar{\delta}\bar{\tau};\alpha} \dot{u}_s^{\bar{\delta}} \dot{u}_s^{\bar{\tau}} - R_{\bar{\delta}\bar{\tau}\sigma}^{\bar{\beta}} (u_s^\sigma - a_s^\sigma) \dot{u}_s^{\bar{\tau}} \dot{u}_s^{\bar{\delta}} + \\ &\quad R_{\bar{\delta}\bar{\tau}\sigma}^{\bar{\beta}} \dot{u}_s^\sigma \dot{u}_s^{\bar{\tau}} u_s^{\bar{\delta}} + \Gamma_{\bar{\nu}\bar{\tau}}^{\bar{\beta}} \Gamma_{\bar{\delta}\bar{\lambda}}^{\bar{\nu}} \dot{u}_s^{\bar{\lambda}} (u_s^{\bar{\delta}} \dot{u}_s^{\bar{\tau}} - u_s^{\bar{\tau}} \dot{u}_s^{\bar{\delta}}). \end{aligned}$$

Proof. The equations (21) and (23) follow immediately from the definition. The remaining proofs are rather computational: To prove (22) we apply D/dt to (21) and differentiate (12) with respect to s . In this way we can eliminate \ddot{u}_s^α . We use (10), and finally we have (22). Observe that we need to consider both ordinary and covariant derivatives of Christoffel symbols. We prove (24) in the same way. \square

In order to describe the variation of the length of closed geodesics in a family, we use the notion of integrating a tensor along a geodesic. Exemplarily we define:

Definition 3.2. *Let $C = C_{\bar{\beta}\bar{\delta}}$ be a tensor on the Kähler manifold X , and γ be a geodesic of length ℓ , parameterized by $u(t) = (u^1(t), \dots, u^n(t))$, such that $\|\dot{u}(t)\| = 1$. Then*

$$\int_\gamma C = \int_\gamma C_{\bar{\beta}\bar{\delta}} dz^{\bar{\beta}} dz^{\bar{\delta}} := \int_0^\ell C_{\bar{\beta}\bar{\delta}}(u(t)) \dot{u}^{\bar{\beta}} \dot{u}^{\bar{\delta}} dt.$$

For covariant tensors of order one this notation coincides with the integration of a differential form along the curve γ . For contravariant tensors the geodesic integral is defined after lowering indices with respect to the metric tensor.

4. FIRST VARIATION OF THE GEODESIC LENGTH FUNCTION

Given a holomorphic family of Kähler-Einstein manifolds with one dimensional base space like in the previous section together with a differentiable family of closed geodesics γ_s with parametrization (13), the length of these is equal to

$$\ell(s) = \int_0^{\ell_0} \|\dot{u}(t, s)\| dt$$

so that

$$(25) \quad \left. \frac{d\ell(s)}{ds} \right|_{s=0} = \frac{1}{2} \int_0^{\ell_0} \frac{d}{ds} \|\dot{u}(t, s)\|^2 dt.$$

We will compute

$$\frac{d}{ds} \|\dot{u}(t, s)\|^2 = \frac{d}{ds} (g_{\alpha\bar{\beta}} \dot{u}^\alpha \dot{u}^{\bar{\beta}}).$$

We denote by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ the inner product with respect to $\omega_{\mathcal{X}}$.

Lemma 4.1. *We have*

$$(26) \quad \frac{d}{ds} (g_{\alpha\bar{\beta}} \dot{u}^\alpha \dot{u}^{\bar{\beta}}) - \frac{d}{dt} \langle u_* \partial_s, \dot{u} \rangle_{\omega_{\mathcal{X}}} = A_{s\bar{\beta}\delta} \dot{u}^{\bar{\beta}} \dot{u}^{\bar{\delta}}.$$

In the computational *proof* one uses (18), (10), (21), and (23).

An immediate consequence of the above Lemma is the following Theorem.

Theorem 4.2 ([A-S, Theorem 3.2]). *Given a holomorphic family of Kähler-Einstein manifolds with negative Ricci curvature, the first variation of the length in a family of closed geodesics γ_s is the geodesic integral of the harmonic Kodaira-Spencer tensors:*

$$(27) \quad \left. \frac{\partial \ell(s)}{\partial s^i} \right|_{s=s_0} = \frac{1}{2} \int_{\gamma(s_0)} A_{i\bar{\beta}\delta} dz^{\bar{\beta}} dz^{\bar{\delta}}.$$

5. SECOND VARIATION OF THE GEODESIC LENGTH FUNCTION

An important function is given by the inner product of harmonic lifts of tangent vectors. In terms of local holomorphic coordinates s^i on S (or coordinates of a smooth ambient space of minimal dimension at a given point of the base) we have:

Definition 5.1. *Let v_i be the horizontal lift of $\partial/\partial s^i$. We put*

$$(28) \quad \varphi_{i\bar{j}} = \langle v_i, v_j \rangle_{\mathcal{X}},$$

where the inner product is taken pointwise.

We list basic properties of the function $\varphi_{i\bar{j}}$ on \mathcal{X} :

$$(29) \quad \varphi_{i\bar{j}} = g_{i\bar{j}} - a_i^\alpha a_{\bar{j}}^{\bar{\beta}} g_{\alpha\bar{\beta}},$$

$$(30) \quad (\square + 1)\varphi_{i\bar{j}} = \langle A_{i\bar{\beta}\delta}, A_{j\bar{\lambda}\tau} \rangle = A_{i\bar{\beta}}^\alpha A_{j\bar{\lambda}}^{\bar{\tau}},$$

$$(31) \quad \int_{\mathcal{X}_s} \varphi_{i\bar{j}} = G_{i\bar{j}}^{\text{WP}},$$

$$(32) \quad \omega_{\mathcal{X}}^{n+1} = \sqrt{-1} \varphi_{i\bar{j}} ds^i \wedge d\bar{s}^{\bar{j}} \wedge \omega_{\mathcal{X}}^n.$$

The first of these equalities follows from the definition. For the second equality cf. [SCH4, Proposition 3]. The equation (31) follows from (30). The last equation (32) is Lemma 6 from [SCH4].

We will apply the following fact:

Theorem ([SCH4]). *The relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$ equipped with the hermitian metric induced by the relative Kähler-Einstein forms is positive, i.e. the matrix $(\varphi_{i\bar{j}})$ is positive definite.*

The lower estimates for $(\varphi_{i\bar{j}})$ from [SCH4] will be applied below.

Again, it is sufficient to do computations for a base space S of dimension one with coordinate s . By abuse of notation, we use s and \bar{s} as indices instead of i and \bar{j} , where i, j can only take the value 1.

Lemma 5.2. *We have*

$$A_{s\bar{\beta}\bar{\delta}} = -\varphi_{s\bar{s};\bar{\beta}\bar{\delta}} - A_{s\bar{\tau}\bar{\beta};\bar{\delta}}\bar{a}_{\bar{s}}^{\bar{\tau}} - A_{s\bar{\tau}\bar{\beta}}\bar{A}_{s\bar{\delta}}^{\bar{\tau}} - A_{s\bar{\tau}\bar{\delta}}\bar{A}_{s\bar{\beta}}^{\bar{\tau}}.$$

Proof. We compute

$$A_{s\bar{\beta}\bar{\delta}} = \left(a_{s\bar{\beta}}\bar{\delta} + a_{s\bar{\tau}}\bar{\Gamma}_{\bar{\beta}\bar{\delta}}^{\bar{\tau}} \right)_{\bar{s}} = a_{s\bar{\beta}}\bar{\delta} + a_{s\bar{\tau}}\bar{\Gamma}_{\bar{\beta}\bar{\delta}}^{\bar{\tau}}.$$

Now the claim follows from (10) and (29). \square

From here we immediately obtain the following identity.

Lemma 5.3. *We have*

$$\begin{aligned} \frac{\partial}{\partial \bar{s}}(A_{s\bar{\beta}\bar{\delta}}\dot{u}^{\bar{\beta}}\dot{u}^{\bar{\delta}}) &= (-\varphi_{s\bar{s};\bar{\beta}\bar{\delta}} - A_{s\bar{\tau}\bar{\beta};\bar{\delta}}\bar{a}_{\bar{s}}^{\bar{\tau}} - 2A_{s\bar{\tau}\bar{\beta}}\bar{a}_{s\bar{\delta}}^{\bar{\tau}})\dot{u}^{\bar{\beta}}\dot{u}^{\bar{\delta}} + \\ &\quad A_{s\bar{\beta}\bar{\delta}}\bar{\Gamma}_{\bar{s}}^{\bar{\tau}}\dot{u}^{\bar{\beta}}\dot{u}^{\bar{\delta}} + A_{s\bar{\beta}\bar{\delta};\gamma}u_s^{\gamma}\dot{u}^{\bar{\beta}}\dot{u}^{\bar{\delta}} + 2A_{s\bar{\beta}\bar{\delta}}\dot{u}_s^{\bar{\beta}}\dot{u}^{\bar{\delta}}. \end{aligned}$$

We need to eliminate mixed derivatives in the parameters t and s . We define a function χ along the geodesics by the formula

$$(33) \quad \chi = \langle A_{s\bar{\lambda}}^{\kappa}\dot{u}^{\bar{\lambda}}\partial_{\kappa}, u_*(\partial_s) \rangle_{\omega_{\chi}} = A_{s\bar{\beta}\bar{\delta}}(u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\bar{\delta}}$$

and obtain

$$\frac{d}{dt}\chi = \frac{D}{dt}(A_{s\bar{\beta}\bar{\delta}})(u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\bar{\delta}} + A_{s\bar{\beta}\bar{\delta}}\frac{D}{dt}(u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\bar{\delta}}.$$

A straightforward calculation using the identities (20) and (21) shows that

$$\begin{aligned} (34) \quad \frac{\partial}{\partial \bar{s}}(A_{s\bar{\beta}\bar{\delta}}\dot{u}^{\bar{\beta}}\dot{u}^{\bar{\delta}}) - 2\dot{\chi} + \frac{d}{dt}(\varphi_{s\bar{s};\bar{\beta}}\dot{u}^{\bar{\beta}}) &= (\varphi_{s\bar{s};\alpha\bar{\beta}} + 2A_{s\bar{\beta}\bar{\delta}}\bar{A}_{s\alpha}^{\bar{\delta}})\dot{u}^{\alpha}\dot{u}^{\bar{\beta}} \\ &\quad - (A_{s\bar{\beta}\bar{\delta};\bar{\tau}}\dot{u}^{\bar{\tau}} + A_{s\bar{\beta}\bar{\delta};\gamma}u_s^{\gamma})(u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\bar{\delta}} \\ &\quad + A_{s\bar{\beta}\bar{\delta};\gamma}\dot{u}^{\bar{\delta}}(u_s^{\gamma}\dot{u}^{\bar{\beta}} - (u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\gamma}). \end{aligned}$$

This concludes the first part of the computation. Altogether we obtained:

Proposition 5.4. *Let $\mathcal{X} \rightarrow S$ be a holomorphic family of Kähler-Einstein manifolds of constant negative Ricci curvature together with a differentiable family of closed geodesics. Then the second variation of the geodesic length function equals*

$$\begin{aligned} (35) \quad \frac{\partial^2 \ell(s)}{\partial s \partial \bar{s}} &= \frac{1}{2} \int_{\gamma_s} \left((\varphi_{s\bar{s};\alpha\bar{\beta}} + 2A_{s\bar{\beta}\bar{\delta}}\bar{A}_{s\alpha}^{\bar{\delta}})\dot{u}^{\alpha}\dot{u}^{\bar{\beta}} \right. \\ &\quad \left. - (A_{s\bar{\beta}\bar{\delta};\bar{\tau}}\dot{u}^{\bar{\tau}} + A_{s\bar{\beta}\bar{\delta};\gamma}u_s^{\gamma})(u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\bar{\delta}} + A_{s\bar{\beta}\bar{\delta};\gamma}\dot{u}^{\bar{\delta}}(u_s^{\gamma}\dot{u}^{\bar{\beta}} - (u_s^{\bar{\beta}} - \bar{a}_{\bar{s}}^{\bar{\beta}})\dot{u}^{\gamma}) \right). \end{aligned}$$

6. SECOND VARIATION OF THE GEODESIC LENGTH FUNCTION ON TEICHMÜLLER SPACES

From now on we assume that fibers of $f : \mathcal{X} \rightarrow S$ are one dimensional. We set $z = z^1$ and also use z and \bar{z} as indices. The preceding formulas and the notation remain valid, if the fibers are equipped with the hyperbolic metric of constant Ricci curvature -1 , i.e. on a fiber \mathcal{X}_s with coordinate function z we have

$$ds^2 = g(z, s) \sqrt{-1} dz \wedge \bar{dz}$$

where $g(z, s)$ satisfies the equation

$$g(z, s) = \frac{\partial^2 \log g(z, s)}{\partial z \partial \bar{z}}.$$

Free homotopy classes of simple closed curves are represented by closed geodesics γ_s with parameterization $u(s, t)$, which depend in a differentiable way upon the parameter s .

According to our general index convention we have $g = g_{z\bar{z}}$. Observe that the harmonic Kodaira-Spencer form

$$A_{s\bar{z}}^z \partial_z \bar{dz}$$

is exactly a harmonic Beltrami differential. Likewise

$$A_{\bar{s}zz}$$

defines a holomorphic quadratic differential. The statement of Proposition 5.4 now reads as follows:

Proposition 6.1. *We have*

$$(36) \quad \frac{\partial^2 \ell(\gamma_s)}{\partial s \partial \bar{s}} = \frac{1}{2} \int_{\gamma_s} \left((\varphi_{s\bar{s}} + g^{\bar{z}z} A_{s\bar{z}\bar{z}} A_{\bar{s}z}^{\bar{z}}) + A_{s\bar{z}\bar{z}} \dot{u}^{\bar{z}} \frac{D}{dt} (u_{\bar{s}}^{\bar{z}} - a_{\bar{s}}^{\bar{z}}) \right).$$

Proof. The term that involves the function φ can be interpreted as a complex Laplacian and (31) is applicable. We use $g \dot{u}^{\bar{z}} = 1$. The harmonicity of the Kodaira-Spencer tensor is equivalent to

$$A_{s\bar{z}\bar{z};z} = 0$$

so that the latter terms in (35) vanish. □

Theorem 6.2. *Let $f : \mathcal{X} \rightarrow S$ be a holomorphic family of hyperbolic Riemann surfaces together with a differentiable family of closed geodesics γ_s . Then*

$$(37) \quad \frac{\partial^2 \ell(\gamma_s)}{\partial s^i \partial \bar{s}^{\bar{j}}} = \frac{1}{2} \int_{\gamma_s} \left((\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) + \left(-\frac{D^2}{dt^2} + 2 \right)^{-1} (A_i) \cdot A_{\bar{j}} \right) \\ + \frac{1}{4\ell(\gamma_s)} \int_{\gamma_s} A_i \cdot \int_{\gamma_s} A_{\bar{j}}.$$

We prove the theorem in Section 8.

Next, we estimate the integrand in (37) from below:

Definition 6.3. *Given any two Hermitian symmetric matrices $M_{i\bar{j}}$ and $N_{i\bar{j}}$, we write $M_{i\bar{j}} \geq N_{i\bar{j}}$, if the difference is a positive semi-definite matrix.*

Corollary 6.4. *We have the inequality*

$$\frac{\partial^2 \ell(\gamma_s)}{\partial s^i \partial s^{\bar{j}}} \geq \frac{1}{2} \left(\int_{\gamma_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) + \frac{1}{\ell(\gamma_s)} \int_{\gamma} A_i \int_{\gamma} A_{\bar{j}} \right)$$

in the sense of Definition 6.3. In particular the geodesic length function is strictly plurisubharmonic.

Again we apply our main theorem from [A-S], which states that

$$\frac{\partial \ell(\gamma_s)}{\partial s_i} = \frac{1}{2} \int_{\gamma_s} A_i$$

and obtain the following statement.

Corollary 6.5. *The logarithm of the geodesic length function is strictly plurisubharmonic: The inequality*

$$\frac{\partial^2 \log \ell(\gamma_s)}{\partial s^i \partial s^{\bar{j}}} \geq \frac{1}{2\ell(\gamma_s)} \int_{\gamma_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) + \frac{1}{4\ell(\gamma_s)^2} \int_{\gamma_s} A_i \int_{\gamma_s} A_{\bar{j}}$$

holds in the sense of Definition 6.3.

A lower estimate for the functions $\varphi_{i\bar{j}} = (\square + 1)^{-1} (A_i \cdot A_{\bar{j}})$ is known:

Proposition (cf. [SCH4]). *There exists a positive function $P_1(d(\mathcal{X}_s))$, which depends on the diameter of \mathcal{X}_s , such that for any solution*

$$(\square + 1)\varphi = \chi,$$

with $\chi \geq 0$ the inequality

$$\varphi(z) \geq P_1(d(\mathcal{X}_s)) \int_{\mathcal{X}_s} \chi g dV$$

holds for all $z \in \mathcal{X}_s$.

The above proposition implies the following estimate, which can be used together with Corollary 6.4 and Corollary 6.5 to obtain further inequalities:

$$\int_{\gamma_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \geq \ell(\gamma_s) \cdot P_1(d(\mathcal{X}_s)) \cdot G_{i\bar{j}}^{\text{WP}}.$$

Lemma 6.6 (cf. [SCH2, Lemma 3]). *Let ℓ_j be positive functions on a complex manifold. Then the following estimate of closed hermitian $(1, 1)$ -forms holds:*

$$\sqrt{-1} \partial \bar{\partial} \log \left(\sum_j \ell_j \right) \geq \frac{1}{\sum_k \ell_k} \sum_j (\ell_j \sqrt{-1} \partial \bar{\partial} \log \ell_j).$$

The above Lemma implies that estimates for the single geodesic length functions carry over to any sum of such functions. Kerckhoff showed in [K] that for a finite number of closed geodesics $\gamma_1, \dots, \gamma_m$, which fill up the Riemann surface the sum of the geodesic length functions provides a proper exhaustion of the Teichmüller space. Wolpert proved in [WO2] that the function $(\ell(\gamma_1) + \dots + \ell(\gamma_m))^{1/2}$ is actually convex along the Weil-Petersson geodesics and $\log(\ell(\gamma_1) + \dots + \ell(\gamma_m))$ is strictly plurisubharmonic (cf. [WO3, WO4]).

Yeung constructs in [Ye] a bounded non-positive strictly plurisubharmonic exhaustion function. His estimates of the second variation of the geodesic length function follow from ours.

Corollary 6.7. *The logarithm of any sum of geodesic length functions is strictly plurisubharmonic with estimates given by Lemma 6.6.*

We conclude the section with an upper estimate, which we state for $\dim S = 1$.

Corollary 6.8. *Let $\|A_s\|_0$ be the maximum of the pointwise norm of the harmonic Beltrami differential taken over the fiber \mathcal{X}_s . Then*

$$(38) \quad \frac{\partial^2 \ell(\gamma_s)}{\partial s \partial \bar{s}} \leq \ell(\gamma_s) \|A_s\|_0^2,$$

$$(39) \quad \frac{\partial^2 \log \ell(\gamma_s)}{\partial s \partial \bar{s}} \leq \frac{3}{4} \|A_s\|_0^2.$$

Proof. The maximum principle applied to the equation (30) yields that

$$\varphi_{s\bar{s}}(z) \leq \|A_s\|_0^2.$$

Furthermore,

$$\int_{\gamma_s} (2 - D^2/dt^2)^{-1} (A_s) \cdot A_{\bar{s}} \leq \frac{1}{2} \int_{\gamma_s} A_s \cdot A_{\bar{s}} \leq \frac{1}{2} \ell(\gamma_s) \|A_s\|_0^2,$$

and finally

$$\left| \int_{\gamma_s} A_s \right|^2 \leq \ell(\gamma_s) \int_{\gamma_s} A_s \cdot A_{\bar{s}}.$$

These estimates imply both inequalities. \square

7. DIFFERENTIAL OPERATORS ALONG CLOSED GEODESICS

When studying covariant differentiation along geodesics $u(t)$ on a fixed Riemann surface, we observe that the obvious identities

$$\frac{D}{dt} \dot{u} = 0 \quad \text{and} \quad \frac{D}{dt} g_{z\bar{z}} = 0$$

can be used to reduce covariant differentiation of tensors along a closed geodesic to the (covariant) differentiation of functions. In our case all functions will be of class C^∞ . Hilbert space theory and regularity theorems are available and need not explicitly be mentioned.

Lemma 7.1. *The operator*

$$L = -\frac{D^2}{dt^2} + 1$$

is invertible with bounded inverse.

Let $\lambda_\nu \geq 0$, $\lambda_0 = 0$ be the eigenvalues of $-\frac{D^2}{dt^2}$. For any function ψ we denote by

$$\psi = \sum_{\nu \geq 0} \psi_\nu$$

the eigenvector decomposition. An inverse of the operator $L - L^{-1}$ is defined on the orthogonal complement C of the kernel of D^2/dt^2 (which is also the kernel of D/dt) with values in the same complement.

Lemma 7.2. *Let*

$$M = (L - L^{-1})^{-1} \circ \left(-\frac{D^2}{dt^2} \right).$$

Then

$$M(\psi) = \sum_{\nu > 0} \left(1 - \frac{1}{2 + \lambda_\nu} \right) \psi_\nu.$$

In particular, $M = 1 - (2 - D^2/dt^2)^{-1}$ on the complement C .

This yields the following estimate.

Proposition 7.3. *For the geodesic integral we have*

$$0 \leq \int_\gamma M(\psi) \bar{\psi} \leq \frac{1}{2} \left(\int_\gamma |\psi|^2 - \frac{1}{\ell(\gamma)} \left| \int_\gamma \psi \right|^2 \right).$$

Proof. By Lemma 7.2 we have

$$0 \leq \int_\gamma M(\psi) \bar{\psi} = \sum_{\nu > 0} \left(1 - \frac{1}{2 + \lambda_\nu} \right) \int_\gamma |\psi_\nu|^2 \leq \frac{1}{2} \int_\gamma (|\psi|^2 - |\psi_0|^2).$$

□

8. PROOF OF THE MAIN THEOREM

We now prove Theorem 6.2. Only the case of $\dim S = 1$ is needed.

For one dimensional fibers (22) and (24) read

$$(40) \quad \frac{D^2}{dt^2}(u_s^z - a_s^z) = (u_s^z - a_s^z) - g_{z\bar{z}} \dot{u}^z \dot{u}^z u_s^{\bar{z}} - A_{s\bar{z};\bar{z}}^z \dot{u}^{\bar{z}} \dot{u}^{\bar{z}},$$

$$(41) \quad \frac{D^2}{dt^2}(u_s^{\bar{z}}) = -g_{z\bar{z}}(u_s^{\bar{z}} - a_s^{\bar{z}}) \dot{u}^z \dot{u}^z + u_s^{\bar{z}}.$$

We define auxiliary functions along the geodesics. Let

$$\begin{aligned} w &= (u_s^z - a_s^z) \dot{u}^{\bar{z}} g_{z\bar{z}}, \\ v &= u_s^{\bar{z}} \dot{u}^z g_{z\bar{z}}, \\ A &= A_{s\bar{z};\bar{z}} (\dot{u}^{\bar{z}})^2. \end{aligned}$$

We apply (40) and (41) and use the notation of the preceding paragraph. The aim is to express the function w in terms of the Kodaira-Spencer form. We have

$$\begin{aligned} Lw &= v + \frac{D}{dt}A, \\ Lv &= w. \end{aligned}$$

Since $(D/dt)A$ is orthogonal to the kernel of D^2/dt^2 , we have

$$\frac{D}{dt}A = (L - L^{-1})w.$$

Now

$$\begin{aligned} \int_{\gamma_s} w \frac{D}{dt} \bar{A} &= \int_{\gamma_s} \left((L - L^{-1})^{-1} \frac{D}{dt}(A) \right) \cdot \frac{D}{dt}(\bar{A}) = \\ &= - \int_{\gamma_s} \left((L - L^{-1})^{-1} \frac{D^2}{dt^2}(A) \right) \cdot \bar{A} = \int_{\gamma_s} M(A) \cdot \bar{A}. \end{aligned}$$

Now Theorem 6.2 follows from Proposition 6.1 and Lemma 7.2. \square

9. WEIGHTED PUNCTURED RIEMANN SURFACES AND CONICAL METRICS

In our previous paper [A-S], we discussed the first variation of the geodesic length function for Teichmüller spaces of weighted punctured Riemann surfaces equipped with hyperbolic conical metrics. Using the extended techniques in [S-T] one can see that our results on second variations and plurisubharmonicity hold true in the conical case (for weights $\geq 1/2$).

REFERENCES

- [A-S] Axelsson, R., Schumacher, G.: Geometric approach to the Weil-Petersson symplectic form. *Comment. Math. Helv.* **85**, 243–257 (2010).
- [E] Elstrodt, J.: Die Resolvente zum Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. Teil 1. *Math. Ann.* **203**, 295–330 (1973).
- [K] Kerckhoff, S.P.: The Nielsen realization problem, *Ann. of Math* **117**, 235–265 (1983).
- [M] McMullen, C.: The moduli space of Riemann surfaces is Kähler hyperbolic. *Ann. of Math.* **151**, 327–357, 2000.
- [SCH1] Schumacher, G.: The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds. Ancona, V. (ed.) et al., *Complex analysis and geometry*. New York: Plenum Press. The University Series in Mathematics. 339–354 (1993).
- [SCH2] Schumacher, G.: Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps. *Math. Ann.* **311**, 631–645 (1998).
- [SCH3] Schumacher, G.: The theory of Teichmüller spaces. A view towards moduli spaces of Kähler manifolds. *Several complex variables VI. Complex manifolds*, *Encycl. Math. Sci.* **69**, 251–310 (1990).
- [SCH4] Schumacher, G.: Positivity of relative canonical bundles for families of canonically polarized manifolds, arXiv:0808.3259v2
- [S-T] Schumacher, G., Trapani, St.: Weil-Petersson geometry for families of hyperbolic conical Riemann Surfaces, preprint (arXiv:0809.0058).

- [SIU] Siu, Y.-T.: Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class. Contributions to several complex variables, Hon. W. Stoll, Proc. Conf. Complex Analysis, Notre Dame/Indiana 1984, Aspects Math. E9, 261–298 (1986).
- [WE] Weil. A.: Final report on contract AF 18(603)-57, Coll. Works (1958).
- [WO1] Wolpert, S.: The Fenchel-Nielsen deformation. Ann. Math. **115**, 501–528 (1982).
- [WO2] Wolpert, S.: Chern forms and the Riemann tensor for the moduli space of curves, Invent. math. **85**, 119–145, (1986).
- [WO2] Wolpert, S.: Geodesic length functions and the Nielsen problem. J. Differ. Geom. **25**, 275–296 (1987).
- [WO3] Wolpert, S.: Convexity of geodesic-length functions: a reprise. In Spaces of Kleinian Groups. Cambridge University Press, 2004 Lond. Math. Soc. Lec. Notes xxx, 1–14, Minsky, Y., Sakuma, M., Series, C. (Eds.)
- [WO4] Wolpert, S.: Behavior of geodesic-length functions on Teichmüller space. J. Diff. Geom. **79**, 277–334 (2008).
- [Ye] Yeung, S.-K.: Bounded smooth strictly plurisubharmonic exhaustion functions on Teichmüller spaces, Math. Res. Lett. **10**, 391–400 (2003).

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